Lecture of Tensor Products and Heisenberg Spin Chain

Multi-Qubit Systems:

If two independents are represented by the states$|\psi_1\rangle, |\psi_2\rangle \in C^2$, then the state of the system made of these two qubits would be $|\psi_1, \psi_2\rangle = |\psi_1\rangle|\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

The products state of the states $|\psi_1\rangle, |\psi_2\rangle \in C^2$ is the pure tensor = $|\psi_1\rangle \otimes |\psi_2\rangle$ it represents the state of a system of two independent qubits in the states $|\psi_1\rangle$ *and* $|\psi_2\rangle$

From our prior definition of tensors, we know that pure tensors are only a generating set of $V \otimes W$, but certain elements of $V \otimes W$ are not pure tensors.
When the state of a 2-qubit system is not in a product state, we call that system to be in an entangled state. This definition will become clearer with the example below.

Example: We can start by looking the state

$|\psi\rangle = \frac{1}{\sqrt{2}}(|(0,0)\rangle + |(1,1)\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \in C^4$

In this case, we can prove that $|\psi\rangle$ is an entangled state. We will star by assuming that $|\psi\rangle$ is in a product state, We let $a_0, a_1, b_0, b_1 \in C$ s.t.

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$
$$|\psi_1\rangle = a_0|0\rangle + a_1|1\rangle$$
$$|\psi_2\rangle = b_0|0\rangle + b_1|1\rangle$$

As a result, we will yield the following,
$|\psi_1\rangle \otimes |\psi_2\rangle = a_0b_0|0\rangle|0\rangle + a_1b_1|1\rangle|1\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|0\rangle$

This result will imply the below constraints:
$a_0b_0 = a_1b_1 = \frac{1}{\sqrt{2}}$ and $a_0b_1 = a_1b_0 = 0$

The first condition contradicts the second condition. Hence, we can conclude that the state is not a product state, but rather, the state is entangled.

Partial Measurements:

Definition: Given $|\psi\rangle = a|0,0\rangle + b|0,1\rangle + c|1,0\rangle + d|1,1\rangle \in C^2 \otimes C^2$

$Measure\ of\ the\ \ 1st\ qubit$:
$1\ with\ probability\ a^2 + b^2 \ --> \ with\ posterior\ state\ = \frac{a|0,0\rangle + b|0,1\rangle}{\sqrt{a^2+b^2}}$
$0\ with\ probability\ c^2 + d^2 \ --> \ with\ posterior\ state\ = \frac{c|1,0\rangle + d|1,1\rangle}{\sqrt{c^2+d^2}}$
$Measure\ of\ the\ \ 2nd\ qubit$:
$1\ with\ probability\ a^2 + c^2 \ --> \ with\ posterior\ state\ = \frac{a|0,0\rangle + c|1,0\rangle}{\sqrt{a^2+c^2}}$
$0\ with\ probability\ b^2 + d^2 \ --> \ with\ posterior\ state\ = \frac{b|0,1\rangle + d|1,1\rangle}{\sqrt{b^2+d^2}}$

Example : We will use the same example state as before in order to practice partial measurements.

$|\psi\rangle = \frac{1}{\sqrt{2}}(|(0,0)\rangle + |(1,1)\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \in C^4$

If we measure the first qubit, we obtain 0 with probability 1/2, and the system is left in the state $|0\rangle|0\rangle$. Then, a measurement of the second qubit yields 0 with probability 1.
However, if we decided to measure the second qubit initially, we would get 0 with probability 1/2. As a result, the measurement of the first qubit of the system has impacted the subsequent measurements of the second qubit.

Spins:

In physics, spin is a fundamental quantum concept, and for our purposes, we'll pragmatically consider it as a linear space representing the su(2) algebra. The discussion won't delve into the detailed physical origin of spin but emphasizes its representation through the specified su(2) algebra commutation relation.

$\left[S^{\alpha}, S^{\beta}\right] = I\epsilon^{\alpha\beta\gamma}S^{\gamma}, \quad \alpha, \beta, \gamma = 1, 2, 3$ where $\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita symbol. The fundamental representation in which the spin operators are given by the Pauli matrices show below are the simplest representation of this algebra:

$S^1 = \frac{1}{2}\sigma^x \;\; S^2 = \frac{1}{2}\sigma^y \;\; S^3 = \frac{1}{2}\sigma^z$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The two basis vectors are below; they are called “spin up” and “spin down”.

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

There exists two useful operators in the fundamental representation:

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These operators act on the basis vectors as below:

$$S^+|\uparrow\rangle = 0, \quad S^-|\uparrow\rangle = |\downarrow\rangle, \quad S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle,$$
$$S^+|\downarrow\rangle = |\uparrow\rangle, \quad S^-|\downarrow\rangle = 0, \quad S^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle.$$

Heisenberg Spin Chain:
A spin chain is essentially a series of spins arranged on a 1-dimensional lattice with L sites. Each site contains a spin, and the interactions between these spins are determined by a specified quantum Hamiltonian.

Hilbert Space:
The Hilbert space of a spin chain is constructed by taking the direct product of linear spaces corresponding to individual spins. Denoted as V, it is expressed as the tensor product of V1 ⊗ V2 ⊗ · · · ⊗ VL, where Vk represents the linear space at site k. This Hilbert space, V, has a dimension of 2L, and a convenient basis is formed by states such as |↑⟩1 ⊗|↑⟩2 ⊗···⊗|↑⟩L and |↓⟩1 ⊗|↑⟩2 ⊗···⊗|↓⟩L. The dimension corresponds to the 2 possible choices (spin-up or spin-down) at each site, and common notation condenses the tensor product symbol for brevity.

Hamiltonian: we've previously discussed a chain of L spins, where the interactions are governed by a Hamiltonian. Specifically, the Hamiltonian for the Heisenberg spin chain is now introduced.

$$\hat{H} = \sum_{n=1}^{L} \left(J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z \right) \ .$$

Special cases:
Firstly, Jx, Jy, Jz are three parameters which specify how strong the spins interact in each direction. We have the following important special cases

1. Jx =Jy =0,Jz≠0. This is the Ising Spin Chain.
2. Jz = 0, Jx = Jy≠ 0. This is the XX spin chain, which is equivalent to a free lattice fermion by Jordan-Wigner transformation.
3. Jx = Jy = Jz≠ 0. This is the isotropic case, which is called the XXX spin chain.
4. Jx = Jy≠ Jz≠ 0. This is the anisotropic case called XXZ spin chain.
5. Jx≠ Jy≠ Jz≠ 0. This is the completely anisotropic case, which is called the XYZ spin chain.

Interacting range: The interaction in the Heisenberg spin chain is characterized by nearest neighbor interactions, where each spin S_n^{α} (α = x, y, z) only interacts with its immediate neighbor Sα on site n+1. This is referred to as nearest neighboring interaction, and the term "interacting range" is defined as the number of sites involved in the Hamiltonian; commonly studied cases have a range of k = 2. However, there's a growing interest in exploring integrable spin chains with larger ranges (k > 2), known as medium or long-range interacting spin chains. It's important to note that the spin operators, denoted as S_n^α, act locally on the spins at site-n without affecting other sites, earning them the label of local spin operators

The XXX Spin Chain:

For the XXX Spin chain, we will modify the Hamiltonian as below:

$$H_{\mathrm{XXX}} = -J\sum^{L}(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z)$$
$$= -\frac{J}{2}\sum_{n=1}^{L}(S_n^- S_{n+1}^+ + S_n^+ S_{n+1}^- + 2S_n^z S_{n+1}^z)$$

and

$$S_n^a = \frac{1}{2}\sigma_n^a, \qquad a = x, y, z$$

Structure of Hilbert space to simplify calculations, the Hilbert space is divided into smaller subspaces based on the number of spin-downs, considering spin-ups as the 'vacuum' and spin-downs as 'excitations.' This perspective becomes clearer in the context of the Bethe ansatz. For a spin chain of length L, the Hilbert space is decomposed into sectors with 0, 1, 2, and so on, spin-downs. For instance, for L = 3, various sectors are formed based on the number of spin-downs.

- Vacuum: $|\uparrow\uparrow\uparrow\rangle$
- One spin-down: $|\downarrow\uparrow\uparrow\rangle$, $|\uparrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$
- Two spin-downs: $|\uparrow\downarrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\rangle$,$|\downarrow\uparrow\downarrow\rangle$
- Three spin-down $|\downarrow\downarrow\downarrow\rangle$